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Explicit results for the correlation length of the finite-sized spherical model of ferromagnetism

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Abstract. We report explicit results for the zero-field correlation length, $\xi(T;L)$, of a spherical-model ferromagnet confined to geometry $L^{d-d'}\times\infty^{d'}$ ($d>2$, $d'\leqslant 2$) and subjected to 'twisted' boundary conditions. In the region of first-order phase transition ($T<T_c$), ξ under 'twisted' boundary conditions obeys the same scaling law as under periodic boundary conditions, though with a different amplitude. In the 'core' region ($T\simeq T_c$), the scaling behaviour changes radically as one moves from one set of boundary conditions to the other—affected greatly by the 'pinning' of the ground-state wavevector k_0 of the system.

1. Introduction

In a recent communication Henkel and Weston (1992) reported an exact calculation of the universal amplitude A of the correlation length ξ of the spherical model of ferromagnetism confined to geometry $L^2\times\infty^1$ and subjected to *antiperiodic* boundary conditions. Invoking a well-known analogy between statistical mechanics and quantum field theory (Henkel 1988, 1990), they derived a relationship between ξ and the thermogeometric parameter y of the system (Pathria 1972, 1983), namely

$$\xi=\tfrac{1}{2}L\left[y^2+\frac{\pi^2}{4}d^*\right]^{-1/2} \tag{1}$$

valid at the bulk critical temperature T_c. Here, L denotes the size of the system in each of the d^* dimensions in which the system is finite while the precise value of y depends on both d^* and the total dimensionality d. Recalling that the corresponding result under periodic boundary conditions is $\xi=L/2y$, we note that the term additional to y^2 in equation (1) is a measure of the 'shift in the quantum levels of the system owing to a switch from periodic to antiperiodic boundary conditions'.

While the main thrust of the Henkel–Weston approach was to extend a certain well-known result of Cardy (1984) for the correlation length of a system from two to three dimensions, we here present an approach that examines the same problem in a considerably generalized context. To begin with, we extend the validity of relationship (1) to all temperature regimes of interest—especially to the region of first-order phase transition ($T<T_c$) where the behaviour of ξ, as a function of L, is radically different from that in the 'core' region ($T\simeq T_c$). Second, we extend the calculation of ξ to a more general dimensionality d which includes the hyperscaling regime ($2<d<4$), the mean-field regime ($d>4$) as well as the borderline dimensionality 4. Third, we introduce a continuous vector parameter $\tau = (\tau_1, \ldots, \tau_{d^*})$ that measures the shift of the quantum levels of the system—such that any $|\tau| > 0$ entails 'twisted' boundary conditions (of which antiperiodic ones, with $\tau_1 = \ldots = \tau_{d^*} = \frac{1}{2}$, are an extreme case), while $\tau = 0$ entails the familiar periodic ones.

In section 2 we provide an independent, theoretical basis for a generalized version of equation (1) by appealing to certain special features of the spin–spin correlation function, $G(R, T; L)$, of the spherical model ferromagnet confined to geometry $L^{d^*} \times \infty^{d'}$ ($d^* + d' = d > 2$). This enables us to determine the correlation length $\xi(T; L)$ of the system, extending the validity of the Henkel–Weston result from $T = T_c$ to all $T \leqslant T_c$; we shall note that the result in question applies at $T \gtrsim T_c$ as well. Of course, for an explicit evaluation of ξ, the parameter y has to be eliminated with the help of the constraint equation of the system (see, for instance, Singh and Pathria 1985). In section 3 this explicit evaluation is carried out for $T < T_c$. We find that, for $d' < 2$, the asymptotic behaviour of ξ under antiperiodic boundary conditions is qualitatively similar to the one found previously under periodic boundary conditions; it is only the amplitudes that differ. For $d' = 2$, the comparison is somewhat subtle. In section 4 we calculate ξ at the bulk critical temperature T_c and find a radically different scaling behaviour depending on whether $|\tau| > 0$ or $|\tau| = 0$; the observed difference is even more striking when d is close to, or greater than, 4. Under 'twisted' boundary conditions the correlation length follows closely the behaviour found in $d = 4 - \varepsilon$ dimensions under *free* (Dirichlet) boundary conditions (Eisenriegler 1985) or in the Ising model with $d > 4$ again under *free* boundary conditions (Rudnick *et al* 1985), and bears little resemblance to the results pertaining to the periodic case. In section 5 we close the paper with some general remarks on the problem studied here.

2. Formulation of the problem

In a separate investigation (Allen and Pathria, unpublished) we have examined the spin–spin correlation function, $G(R, T; L)$, of a d-dimensional, field-free spherical-model ferromagnet confined to geometry $L^{d^*} \times \infty^{d'}$ ($d^* + d' = d > 2$) under 'twisted' boundary conditions. Using techniques developed by Joyce (1972), by Barber and Fisher (1973) and by Singh and Pathria (1987), we find that for $R, L \gg a$ where a is the lattice constant of the system

$$G(\boldsymbol{R}, T; L) \approx \frac{T}{4\pi^{d/2} J} \left(\frac{a}{L}\right)^{d-2} \sum_{\boldsymbol{q}(d^*)} \prod_{j=1}^{d^*} \cos(2\pi q_j \cdot \tau_j) \left[\frac{y}{(|\boldsymbol{q} + \boldsymbol{\varepsilon}_\perp|^2 + \varepsilon_\parallel^2)^{1/2}}\right]^{(d-2)/2}$$

$$\times K_{(d-2)/2}(2y(|\boldsymbol{q} + \boldsymbol{\varepsilon}_\perp|^2 + \varepsilon_\parallel^2)^{1/2}) \qquad (2)$$

with

$$\boldsymbol{\varepsilon}_\perp = \boldsymbol{R}_\perp / L \qquad \varepsilon_\parallel = R_\parallel / L. \qquad (3)$$

Here, J is the nearest-neighbour interaction parameter, τ is a continuous vector parameter with components $\tau_1, \ldots, \tau_{d^*}$ each lying in the interval $(0, \frac{1}{2})$, $\boldsymbol{R}_\perp$ is the component of $\boldsymbol{R}$ in the d^*-dimensional sub-space while $\boldsymbol{R}_\parallel$ $(= \boldsymbol{R} - \boldsymbol{R}_\perp)$ is the corresponding component in the d'-dimensional sub-space. The sum involving modified Bessel functions $K_\nu(z)$ goes over the entire $\boldsymbol{q}$-space in d^* dimensions, while the parameter y is a scaled variable defined by the relation

$$y=\tfrac{1}{2}\frac{L}{a}\sqrt{\phi} \qquad \left[\phi=\frac{\lambda}{J}-2d\right] \tag{4}$$

λ being the (spherical) field that enables one to satisfy the constraint on spins appropriate to the model under study. This leads to the constraint equation

$$\left(\frac{1}{T}-\frac{1}{T_c}\right)\approx\frac{1}{4\pi^{(4-d)/2}J}\left(\frac{a}{L}\right)^{d-2}Q_\tau\left(\frac{d-2}{2}\middle|d^*;y\right) \qquad 2<d<4 \tag{5}$$

where

$$Q_\tau(\nu\,|\,d^*;y)=\left(\frac{y^2}{\pi^2}\right)^{\nu}\left[\sum_{q(d^*)}{}' \cos(2\pi\boldsymbol{q}\cdot\boldsymbol{\tau})\,\frac{K_\nu(2yq)}{(yq)^\nu}+\tfrac{1}{2}\Gamma(-\nu)\right]. \tag{6}$$

As shown elsewhere (Allen and Pathria 1993a—hereafter referred to as I), the function $Q_\tau(\nu\,|\,d^*;y)$ is regular in y^2 for all $y^2>-\pi^2\tau^2$—a fact vitally important for analyses pertaining to 'twisted' boundary conditions under which the minimum value of ϕ, and hence of y^2, is negative; in fact, $(y^2)_{\min}$ is precisely equal to $-\pi^2\tau^2$ and is attained at $T=0$ K.

The problem of y^2 becoming negative is seemingly detrimental to the sum appearing in equation (2) for $G(\boldsymbol{R}, T; L)$ which is initially defined only for $y>0$. An application of the Poisson summation formula, however, renders this expression into the form

$$G(\boldsymbol{R}, T; L)\approx\frac{T}{4\pi^{d'/2}J}\left(\frac{a}{L}\right)^{d-2}\sum_{n(d^*)}\prod_{j=1}^{d^*}\cos[2\pi(n_j+\tau_j)\cdot\varepsilon_j]$$
$$\times\left[\frac{\varepsilon_\parallel}{(y^2+\pi^2|\boldsymbol{n}+\boldsymbol{\tau}|^2)^{1/2}}\right]^{(2-d')/2}K_{(2-d')/2}\left(2\varepsilon_\parallel(y^2+\pi^2|\boldsymbol{n}+\boldsymbol{\tau}|^2)^{1/2}\right) \tag{7}$$

which may be analytically continued into the region $y^2<0$—right up to, but excluding, the point $y^2=-\pi^2\tau^2$ where the true singularity of the problem lies. In terms of temperature, expression (7) is valid down to, but excluding, $T=0$ K.

For $y\gg1$ (which pertains to the region $T\gtrsim T_c$), the sum in (7) may be approximated by an integral over $\boldsymbol{n}(d^*)$, leading to the bulk result

$$G(\boldsymbol{R}, T; L)=\frac{T}{2(2\pi)^{d/2}J}\left(\frac{a^2}{\xi_B R}\right)^{(d-2)/2}K_{(d-2)/2}\left(\frac{R}{\xi_B}\right) \tag{8}$$

$$\approx\frac{Ta^{d-2}}{4J(2\pi R)^{(d-1)/2}\xi_B^{(d-3)/2}}\,\mathrm{e}^{-R/\xi_B} \qquad R\gg\xi_B \tag{9}$$

where $R=(R_\perp^2+R_\parallel^2)^{1/2}$, while $\xi_B\approx L/2y=a/\sqrt{\phi}$ is the bulk correlation length. Clearly, there is no ambiguity in defining ξ in the region $T\gtrsim T_c$.

As T decreases from above and approaches the close vicinity of T_c, y^2 under 'twisted' boundary conditions may become negative, with magnitude of order unity. As T decreases further, y^2 changes fast to become almost equal to $-\pi^2\tau^2$ and stays so until $T\to0$ K and $y^2\to-\pi^2\tau^2$. The situation throughout this region can be handled pretty well through expression (7). In particular, keeping $\varepsilon_\perp$ fixed and letting $\varepsilon_\parallel$ become increasingly large, the dominant contribution to the sum in (7) comes from the term with $\boldsymbol{n}=0$, with the result†

$$G(\boldsymbol{R}, T; L) \approx \frac{T}{4\pi^{d'/2}J}\left(\frac{a}{L}\right)^{d-2} \prod_{j=1}^{d^*} \cos(2\pi\tau_j \cdot \varepsilon_j)\left[\frac{\varepsilon_\parallel}{(y^2+\pi^2\tau^2)^{1/2}}\right]^{(2-d')/2} \times K_{(2-d')/2}(2\varepsilon_\parallel(y^2+\pi^2\tau^2)^{1/2}). \tag{10}$$

With $G(\boldsymbol{R})$ factorized in terms of the components $\boldsymbol{R}_\perp$ and $\boldsymbol{R}_\parallel$, we may invoke the correlation function of the d'-dimensional *bulk* system, namely

$$G(\boldsymbol{R}, T; \infty^{d'}) = \frac{T}{2(2\pi)^{d'/2}J}\left(\frac{\xi R}{a^2}\right)^{(2-d')/2} K_{(2-d')/2}\left(\frac{R}{\xi}\right) \tag{11}$$

and write (10) in the highly instructive form

$$G(\boldsymbol{R}, T; L^{d^*}\times\infty^{d'}) \approx \left(\frac{a}{L}\right)^{d^*} \prod_{j=1}^{d^*} \cos\left[2\pi\frac{\tau_j \cdot \boldsymbol{R}_j}{L}\right] G(\boldsymbol{R}_\parallel, T; \infty^{d'}) \qquad R_\perp, L \ll R_\parallel, \xi \tag{12}$$

with

$$\xi = \frac{L}{2(y^2+\pi^2\tau^2)^{1/2}}. \tag{13}$$

The interpretation of this result is straightforward. In particular, we note that in the perceived situation the parameters $R_\perp$ and L are much smaller than $R_\parallel$ and ξ, with the result that, while the decay of correlations in the direction of $\boldsymbol{R}_\perp$ is determined solely by the 'twist' parameter τ, that in the direction of $\boldsymbol{R}_\parallel$ is determined by a (correlation) function pertaining in form to a d'-dimensional bulk system but scaled by a length ξ given by (13). Clearly, ξ represents the correlation length of the actual system in geometry $L^{d^*}\times\infty^{d'}$ and *not* of the d'-dimensional bulk system (which would have nothing to do with L).

Comparing (13) with (1) and remembering that, with antiperiodic boundary conditions applied along all finite dimensions, $\tau^2 = \frac{1}{4}d^*$, we find that our expression for ξ, which should be valid for all $T < T_c$, is precisely the same as that derived by Henkel (1988) for $T = T_c$. The validity of Henkel's formula (1) is thus extended to all $T \leqslant T_c$. At the same time, since for $y \gg 1$ this result reduces to the τ-independent expression $L/2y$, it applies at $T \gtrsim T_c$ as well.

In passing, we note that relationship (13) for the correlation length of the spherical model under *antiperiodic* boundary conditions had been conjectured earlier by Singh *et al* (1986).

3. Correlation length at $T < T_c$

In the region of first-order phase transition ($T < T_c$), where $y^2 \simeq -\pi^2\tau^2$, we may express the various functions of y in terms of the small parameter $(y^2+\pi^2\tau^2)$. Using constraint

† Equivalent contributions arise from $n_j = 0$ and $n_j = -1$ if $\tau_j = \frac{1}{2}$. Thus, depending on the precise structure of τ, we must include a multiplicity factor g_τ in (10); see, for instance, equations (14), (19), etc. The omission of this factor does not, however, affect the argument leading to the desired relationship (13).

equation (5), we should then be able to determine $(y^2+\pi^2\tau^2)$ as a function of T and L, and hence, by (13), $\xi(T;L)$. To accomplish this task we make use of the asymptotic formula (see I).

$$\lim_{y^2\to-\pi^2\tau^2} Q_\tau(\nu\,|\,d^*;y)\approx\tfrac{1}{2}g_\tau\pi^{\frac{1}{2}d^*-2\nu}\Gamma(\tfrac{1}{2}d^*-\nu)(y^2+\pi^2\tau^2)^{\nu-\frac{1}{2}d^*}\qquad \nu<\tfrac{1}{2}d^* \tag{14}$$

where g_τ denotes the multiplicity of the terms, in the sum over $\boldsymbol{n}(d^*)$, for which $|\boldsymbol{n}+\boldsymbol{\tau}|=\tau$. In general, $g_\tau=2^r$ where $r(\leqslant d^*)$ is the number of components τ_j, of $\boldsymbol{\tau}$, that equal $\frac{1}{2}$; for each of these components, *two* terms (with $n_j=0$ and -1) contribute equally toward the sum. If antiperiodic boundary conditions are applied in each of the d^* finite dimensions, then $g_\tau=2^{d^*}$.

Substituting (14) into (5), we obtain

$$(y^2+\pi^2\tau^2)\approx\left[\frac{g_\tau\Gamma[(2-d')/2]}{4\pi^{d'/2}z_1}\right]^{2/(2-d')}\qquad d'<2 \tag{15}$$

where z_1 is the scaled variable appropriate to this region (see Fisher and Privman 1985, Privman 1990)

$$z_1=\Upsilon(T)L^{d-2}/T \tag{16}$$

$\Upsilon(T)$ being the helicity modulus of the bulk system which, for a spherical-model ferromagnet, is known to be (Fisher *et al* 1973)

$$\Upsilon(T)=\frac{2J}{a^{d-2}}\left(1-\frac{T}{T_c}\right). \tag{17}$$

Expression (13) then gives

$$\xi_{AP}(T;L)\approx\left[\frac{(4\pi)^{d'/2}\Upsilon(T)}{\Gamma[(2-d')/2]T}\right]^{1/(2-d')}\left(\frac{L}{2}\right)^{(d-d')/(2-d')}\qquad d'<2. \tag{18}$$

Comparing (18) with the corresponding expression for the periodic case (Singh and Pathria 1987), we find that in this region the two results are formally the same, except for a difference in amplitudes resulting from the replacement of L by $\frac{1}{2}L$ as one switches from the periodic boundary conditions to the antiperiodic ones. Further studies show that, so long as $d'<2$, expression (18) for $\xi(T;L)$ at $T<T_c$ holds for $d\geqslant 4$ as well.

For $d'=2$, the situation is formally different. Now we employ the asymptotic formula

$$\lim_{y^2\to-\pi^2\tau^2} Q_\tau(\nu\,|\,d^*;y)\approx\tfrac{1}{2}g_\tau\pi^{-\frac{1}{2}d^*}\ln\frac{1}{y^2+\pi^2\tau^2}+M_\tau(d^*)\qquad \nu=\tfrac{1}{2}d^* \tag{19}$$

where $M_\tau(d^*)$ is a constant defined in I. We now obtain

$$\xi_\tau(T;L)\approx\frac{L}{2}\exp\left[\frac{2\pi}{g_\tau}\left\{z_1-\frac{M_\tau(d-2)}{2\pi^{(4-d)/2}}\right\}\right]\qquad d'=2. \tag{20}$$

Thus, the quantity ξ/L in this case varies *exponentially* with the scaled variable z_1, the constant $M_\tau(d^*)$ affecting only the amplitude. For $d=3$ (and hence $d^*=1$)

$$M_0(1)=-\pi^{-1/2}\ln 2\qquad M_{1/2}(1)=\pi^{-1/2}\ln(\pi/2) \tag{21, 22}$$

leading to results that agree with the previous calculations of Singh and Pathria (1985, 1987). In the present case, the amplitude of ξ differs qualitatively—depending on whether d is less than 4 as above or not. For $d=4$, for instance, we obtain

$$\xi_\tau(T;L)\approx\frac{L}{2}\left(\frac{2\pi a}{L}\right)^{\pi\tau^2/g_\tau}\exp\left[\frac{2\pi}{g_\tau}\left\{z_1-\frac{M_\tau(2)}{2}+\frac{\psi(2)-|C_4|}{4}\tau^2\right\}\right], \tag{23}$$

where $\psi(2)$ is the digamma function, $C_4\simeq-4.7920$, while $M_\tau(2)$ can be obtained from I; as $\tau\to0$, the result of a recent study pertaining to the periodic case (Singh and Pathria 1992) is recovered. For $d>4$, on the other hand,

$$\xi_\tau(T;L)\approx\frac{L}{2}\exp\left[\frac{2\pi}{g_\tau}\left\{z_1-\frac{(2\pi\tau)^2}{v_3}-\frac{\pi^{(d-4)/2}M_\tau(d-2)}{2}\right\}\right] \tag{24}$$

where v_3 is another scaled variable defined by (see Singh and Pathria 1988)

$$v_3=w^{-1}\left(\frac{a}{L}\right)^{d-4}\qquad w=\tfrac{1}{4}\int_0^\infty[\mathrm{e}^{-x}I_0(x)]^d x\,\mathrm{d}x. \tag{25}$$

The constant $M_\tau(d-2)$ in this case has to be determined numerically.

4. Correlation length at $T=T_c$

There have been numerous studies on how the correlation length ξ varies explicitly with L (and d) at the bulk critical temperature T_c for finite-size $O(n)$ models under periodic boundary conditions (see, for instance, Brézin 1982, Brézin and Zinn-Justin 1985, Luck 1985, Rudnick *et al* 1985b). However, only limited results are available for systems under non-periodic boundary conditions, such as Eisenriegler (1985), Henkel (1988), Henkel and Weston (1992). Here we provide results for the correlation length $\xi(T_c;L)$ of the spherical model under 'twisted' boundary conditions in general geometry $L^{d-d'}\times\infty^{d'}$ for several regimes of d.

Case 1: $2<d<4$

The spherical constraint (5) now reduces to the condition

$$Q_\tau\left(\frac{d-2}{2}\middle|d^*;y\right)=0$$

which, for $2<d<4$, may be re-written as (see I)

$$D_\tau(\nu\,|\,d^*)+\Gamma(\tfrac{1}{2}d^*-\nu)\sum_{n(d^*)}\left\{\left[|n+\tau|^2+\frac{y^2}{\pi^2}\right]^{\nu-\frac{1}{2}d^*}-|n+\tau|^{2\nu-d^*}\right\}=0$$
$$\nu=(d-2)/2 \tag{26}$$

where $D_\tau(\nu\,|\,d^*)$ is another constant defined in I. Using a generalized definition of $D_\tau(\nu\,|\,d^*)$, equation (26) can also be written as

$$\sum_{l=0}^{\infty}\frac{D_\tau(\nu-l\,|\,d^*)}{l!}\left(-\frac{y^2}{\pi^2}\right)^l=0 \tag{27}$$

effectively reducing the problem from one involving a d^*-dimensional sum over vector $\boldsymbol{n}$ to one involving a 1-dimensional sum over scalar l; of course, a knowledge of the constants $D_\tau(\nu-l\,|\,d^*)$ is essential for this reduction.

For $d=3$ and $d^*=1$, the problem can be solved exactly, for in this case equation (26) assumes the form

$$2\ln(2\sin\pi\tau)+\ln\prod_{n=-\infty}^{\infty}\left(1+\frac{y^2}{\pi^2(n+\tau)^2}\right)=0. \tag{28}$$

The infinite product in (28) can be written in a closed form (see Gradshteyn and Ryzhik 1980) which leads to the explicit solution

$$y_\tau\,|_{T=T_c}=\cosh^{-1}(\tfrac{5}{4}-\sin^2\pi\tau)^{1/2}. \tag{29}$$

Special values, $y_0=\ln(\sqrt{5}+1)/2$ and $y_{1/2}=\mathrm{i}\pi/3$, are already known; the intermediate values, $y_{1/6}=0$, $y_{1/4}=\mathrm{i}\pi/6$ and $y_{1/3}=\mathrm{i}\pi/4$, may be noted now.

For $d^*=2$, equation (2) cannot be written in a closed form; the constants $D_\tau(\omega\,|\,2)$ are, however, known for $\tau=(\frac{1}{2},\frac{1}{2})$, viz.

$$D_\tau(\omega\,|\,2)=4.2^{1-\omega}\Gamma(1-\omega)\lambda(1-\omega)\beta(1-\omega) \tag{30}$$

where $\lambda(s)$ and $\beta(s)$ are analytic continuations of the functions

$$\lambda(s)=(1-2^{-s})\zeta(s)=\sum_{l=0}^{\infty}(2l+1)^{-s}\qquad s>1 \tag{31}$$

$$\beta(s)=\sum_{l=0}^{\infty}(-1)^l(2l+1)^{-s}\qquad s>0. \tag{32}$$

Equation (27) then reduces to

$$\sum_{l=0}^{\infty}\binom{\nu-1}{l}\lambda(l+1-\nu)\beta(l+1-\nu)\left(\frac{2y^2}{\pi^2}\right)^l=0 \tag{33}$$

which may be solved numerically for y. For $d=3$, i.e. $\nu=\frac{1}{2}$, we obtain: $y\simeq\mathrm{i}\,1.25213$ which may be compared with the value, $\mathrm{i}\,5/4$, conjectured by Henkel and Weston (1992); see also Allen and Pathria (1993b).

For $d=d^*=3$, equation (26) assumes the form

$$D_\tau(\tfrac{1}{2}\,|\,3)-\frac{y^2}{\pi^2}\sum_{\boldsymbol{n}(3)}\frac{1}{|\boldsymbol{n}+\boldsymbol{\tau}|^2[|\boldsymbol{n}+\boldsymbol{\tau}|^2+y^2/\pi^2]}=0 \tag{34}$$

which can also be rendered into a form similar to (27). The three-dimensional sums appearing in the definitions of the constants $D_\tau(\frac{1}{2}-l\,|\,3)$ can be simplified considerably by using methods developed earlier (see, for instance, Chaba and Pathria 1976), with the result that $y_{\mathrm{AP}}\simeq\mathrm{i}\,1.3267$. Combining the foregoing results, we may write for the three-dimensional spherical model under antiperiodic boundary conditions

$$\frac{\xi_c}{L}\approx\begin{cases}\dfrac{3}{\sqrt{5}\pi}=0.4271 & d^*=1 \qquad (35a)\\ 0.2725 & d^*=2 \qquad (35b)\\ 0.2105 & d^*=3 \qquad (35c)\end{cases}$$

As $d \to 2_+$, equation (26) becomes analytically tractable. We note that, as $\nu = (d-2)/2 \to 0_+$, the constant $D_\tau(\nu \mid d^*)$ approaches the limiting value $-\pi^{d^*/2}\nu^{-1}$; again, see I. The small quantity $(y^2 + \pi^2\tau^2)$ is then determined by the leading term(s) of the sum over $\boldsymbol{n}(d^*)$, with the result

$$\frac{\xi_c}{L} \approx \frac{1}{2\sqrt{\pi}} \left[\frac{1}{2} g_\tau \Gamma\left(\frac{2-d'}{2}\right)(d-2) \right]^{-1/(2-d')} \tag{36}$$

which may be compared with the corresponding result of Singh and Pathria (1987) under periodic boundary conditions. For $d' = 1$, the foregoing result reduces to

$$\frac{\xi_c}{L} \approx \frac{1}{\pi g_\tau (d-2)} \tag{37}$$

which invites comparison with the conformal-invariance predictions for $(L^1 \times \infty^1)$-strips (Cardy 1984), namely

$$\frac{\xi_c}{L} \approx \begin{cases} \dfrac{1}{2\pi x_\sigma} & \tau = 0 \quad (38a) \\[2ex] \dfrac{1}{4\pi x_\sigma} & \tau = \frac{1}{2} \quad (38b) \end{cases}$$

where $x_\sigma = \frac{1}{2}\eta$, η being the well-known critical exponent. A generalization of (38) to arbitrary d makes $x_\sigma = \frac{1}{2}(d - 2 + \eta)$ (Cardy 1987), which, for the spherical model, reduces to $x_\sigma = \frac{1}{2}(d-2)$. Remembering that g_τ under periodic boundary conditions is equal to 1 while under antiperiodic boundary conditions it is 2, comparison between the two sets of results becomes truly striking.

As $d \to 4_-$, the value of y^2 is essentially determined by the first two terms of (27). We now obtain

$$y^2 \simeq \tfrac{1}{2}\varepsilon \sum_{\boldsymbol{q}(d^*)}{}' \frac{\cos(2\pi \boldsymbol{q} \cdot \boldsymbol{\tau})}{q^2} \qquad \varepsilon = (4-d) \ll 1. \tag{39}$$

Clearly, $y^2 \ll 1$ and hence

$$\xi_c/L \approx 1/2\pi\tau \tag{40}$$

valid for any value of d', so long as $\tau > 0$. It follows that, as $d \to 4_-$, the scaling behaviour of ξ is significantly affected by the boundary conditions imposed on the system. In passing, we note that the parameter y^2 now provides only a small, of order ε, correction to the main result (40). In contrast, the corresponding result under periodic boundary conditions $(\tau \to 0)$ is

$$\frac{\xi_c}{L} \approx \begin{cases} \dfrac{1}{2\sqrt{\pi}} \left[\dfrac{1}{2} \Gamma\left(\dfrac{2-d'}{2}\right)\varepsilon \right]^{-1/(4-d')} & d' < 2 \quad (41a) \\[2ex] [2\pi\varepsilon \ln(1/\varepsilon)]^{-1/2} & d' = 2. \quad (41b) \end{cases}$$

Case 2: d=4

At the upper critical dimension $d=d_>=4$, the spherical constraint at $T=T_c$ and for $\tau>0$ is again satisfied by an infinitesimally small value of y^2, namely

$$y^2 \simeq \frac{1}{2\ln(L/a)} \sum_{q(d^*)}' \frac{\cos(2\pi \boldsymbol{q}\cdot\boldsymbol{\tau})}{q^2}. \tag{42}$$

The correlation length ξ_c is, therefore, given by equation (40) as before. The corresponding result under periodic boundary conditions is given by Singh and Pathria (1992) and, quite expectedly, bears no resemblance to (40).

Case 3: d>4

The relevant value of y^2 is now determined by the implicit equation

$$y^2 \approx \tfrac{1}{8}\pi^{(d-4)/2} v_3 Q_\tau\left(\frac{d-2}{2}\middle| d^*; y\right) \qquad 4<d<6 \tag{43}$$

where v_3 is the scaled variable given in (25). Clearly, y^2 continues to be small in value and is approximately given by

$$y^2 \simeq \Gamma\left(\frac{d-2}{2}\right) \frac{v_3}{16\pi^{d/2}} \sum_{q(d^*)}' \frac{\cos(2\pi \boldsymbol{q}\cdot\boldsymbol{\tau})}{q^{d-2}}. \tag{44}$$

It follows that, once again, ξ_c is given by equation (40), with y^2 providing only a small correction.

5. Concluding remarks

The results reported in this paper provide a broad generalization of the ones reported by previous authors on the correlation length $\xi(T; L)$ of a finite-sized spherical model under antiperiodic boundary conditions. Through a detailed analytical study subject to 'twisted' boundary conditions, of which both periodic and antiperiodic are special cases, we have explored the region of first-order phase transition ($T<T_c$) as well as the core region exemplified by the bulk critical point $T=T_c$. The geometry considered here is also more general, namely $L^{d-d'} \times \infty^{d'}$, with $d' \leqslant 2$ and $2<d<6$.

Finite-size scaling in the region $T<T_c$, where $\xi \gg L$, is formally the same for all $\tau \geqslant 0$, so long as $d'<2$; any difference arising from a 'twist' in the ground-state spin-wave mode(s) appears in the amplitude only. Of course, the general rule of having L replaced by $\frac{1}{2}L$, whenever boundary conditions along the finite dimensions of the system are replaced by antiperiodic ones, seems quite central to the analysis presented here; this feature is also tied closely to the multiplicity factor g_τ appearing in the various expressions obtained here. We suspect that these observations apply to all $O(n)$ models with $n \geqslant 2$. For $d'=2$, the dependence of ξ/L on L becomes exponential but the amplitudes now differ in form depending on whether d is less than, equal to, or greater than 4.

At the bulk critical temperature ($T=T_c$), we find a significantly distinct scaling behavior between the 'twisted' and 'straight' (or periodic) boundary conditions—particularly as it applies to systems near and above the upper critical dimension. In the former case, for any $\tau > 0$, one finds a fluctuation-irrelevant (mean-field) critical behaviour (Ma 1976), i.e. ξ_c is pinned by the lowest allowed wavevector $k_0^{-1} \approx L/2\pi\tau$ (see equation (40)), with small corrections in the variables $(4-d)$, $1/\ln(L/a)$ or v_3, depending on whether $d \lesssim 4$, $d=4$ or $d>4$ (see equations (39), (42) and (44) for y^2). In the latter case ($\tau = 0$), fluctuations in the system dominate at $T = T_c$, leading to a singular dependence of ξ_c on these very variables (see equations (41) and the corresponding results of Singh and Pathria (1988, 1992) for $d \geqslant 4$). A common feature in either case, irrespective of the pinning involved or whether d' is less than or equal to 2, is that it is the parameter y^2 (and *not* the correlation length ξ_c itself) that is determined by the replacement $8\pi^2 w \to 1/(d-4) \to \ln(L/a) \to 1/(4-d)$ as the total dimensionality of the system decreases from $d>4 \to d \gtrsim 4 \to d=4 \to d \lesssim 4$. More general conclusions along these lines, applicable to all $n \geqslant 2$, are not fully clear at this point.

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